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Ramanujan relations of higher exponents: $a^n + b^n = c^n + d^n$

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ABSTRACT

In my paper published in April-Edition, 2014, vol-5 of IJSER it was shown how to recognize a number whether it is capable of producing Ramanujan relations or not? If capable, how many wings does the number have? How to establish all those relations? We observed that everything depends upon Ramanujan factor. But the different angles of R-factor and R-relations were missing to be highlighted. Moreover, what happens when exponent goes beyond three? In this paper I would show the nature of accepted integers of higher exponents and the general analysis of those exponents by theory of algebraic equation of single variable.

Keywords

Ideal Ramanujan number, Ramanujan factor, Ramanujan number, Ramanujan relation, Significant & insignificant group, Wing

1. Introduction

For the existence of Ideal Ramanujan Number of higher exponents multiple of three i.e. $a^{3\alpha} + b^{3\alpha} = c^{3\alpha} + d^{3\alpha} = \ldots\ldots$ if the corresponding R_fs are $12xy - 3z_i^2$ (i = 1, 2, 3, ….) then obviously $z_1 = a^\alpha + b^\alpha$, $z_2 = c^\alpha + d^\alpha$ & so on.
This z_i can be said as Key factors (k_f) of higher exponents multiple of three.
$\Rightarrow N = a^{2n+1} + b^{2n+1} = c^{2n+1} + d^{2n+1} = \ldots\ldots$, n > 1 is also the product of prime numbers only so far ideal numbers are concerned and $p^{2n+1}.N$ may produce Ramanujan numbers of higher exponents like n = 1.

Now the question is whether there exists any IRN of exponent three where K_fs are found to be in the form of $a^{2n+1} + b^{2n+1}$ (n > 1) at least in two cases.

2. Any IRN relation of two wings is obtained by p^3.(IRN of single wing) where p = 2 or 3 only.

Say, $N = e_1^3 + o_1^3 = e_2^3 + o_2^3$ where e & o denotes even & odd number respectively and
$\gcd(e_1, o_1) = \gcd(e_2, o_2) = 1$.
e_1 & o_1 are the roots of nature ½.(A ± B) where one must be divisible by 3 & one must be divisible by 2 (may be on the same element or may not be). Similar case is for e_2 & o_2 also. So 3 will be a common factor in between two elements on either side, say in between e_1 & o_2.
$\Rightarrow 3^3(e_3^3 - o_3^3) = e_2^3 - e_1^3$.
If $(e_3^3 - o_3^3) = N_1$ then $3^3.N_1$ must produce one significant R_f in the form of square integer so as to receive the relation $(3e_3)^3 - (3o_3)^3 = e_2^3 - o_1^3 \Rightarrow e_1^3 - o_2^3 = e_2^3 - o_1^3 \Rightarrow e_1^3 + o_1^3 = e_2^3 + o_2^3$.
Here, Obviously $\gcd(e_3, o_3) = 1$.
Because, our initial relation was $N_1 = e_1^3 + o_1^3 = e_2^3 + o_2^3$ where $\gcd(e_1, o_1) = \gcd(e_2, o_2) = 1$ & we received roots (e_1, o_1) & (e_2, o_2) by its corresponding R_fs. Same relation can be written as $3^3\{e_3^3 + (-o_3)^3\} = e_2^3 + (-o_1)^3 = N_2$ where by same procedure we received the roots $(e_3, -o_3)$ & $(e_2, -o_1)$ by its corresponding R_fs.
Hence, $\gcd(e_3, o_3) = 1$. Similarly, $\gcd(e_1, e_2) = 2$
[Please refer my paper published in April edition. vol-5, 2014 of IJSER, heading 5. It is true only for p = 3, 2]

Finally we can say, an ideal Ramanujan relation is of the form $(3e_1)^3 + o_1^3 = e_2^3 + (3o_2)^3$ where e & o denote even & odd numbers respectively & $\gcd(e_1, o_1) = \gcd(e_2, o_2) = \gcd(e_1, o_2) = \gcd(o_1, o_2) = 1$. Also, $\gcd(e_1, e_2) = 2$
Or, it is of the form $(e_1)^3 + (3o_1)^3 = (e_2)^3 + (3o_2)^3$ where similar gcd rules are applicable by their absolute values.

3. $a^{3\alpha} + b^{3\alpha} = c^{3\alpha} + d^{3\alpha}$ has no existence for α > 1

It is quite obvious that 3^3.[(3^3.IRN) of maximum wings] cannot produce any significant or additional relations as because there is no any significant change of R_f for square integer. All search of R_f as square integer are already made completed during first time multiplication of 3^3 and we have received maximum wings. So ($3^3.3^3.3^3$……. α times). (3^3.IRN of maximum wings) i.e. $3^{3\alpha}$.(3^3.IRN of maximum wings) cannot produce any significant or additional relations. All the elements of (3^3.IRN of maximum wings) will be multiplied by 3^α only.
So if $a^{3\alpha} + b^{3\alpha} = c^{3\alpha} + d^{3\alpha}$ exists it will exist in the form of $(3a)^{3\alpha} + b^{3\alpha} = (3c)^{3\alpha} + d^{3\alpha}$
i.e. in the form of $3^{3\alpha}(a^{3\alpha} - c^{3\alpha})$ which has no significant meaning for α > 1. It originates from $3^3\{(a^\alpha)^3 - (c^\alpha)^3\}$ which may produce relations like $(3a^\alpha)^3 - (3c^\alpha)^3 = \ldots\ldots$ which is completely different from $a^{3\alpha} + b^{3\alpha} = c^{3\alpha} + d^{3\alpha}$.
Similar logic can be given with respect to 2^α also.
Hence, $a^{3\alpha} + b^{3\alpha} = c^{3\alpha} + d^{3\alpha}$ has no existence for α > 1

4. Few examples with respect to $3^3.(a^3 - b^3)$ & $2^3.(a^3 - b^3)$ both.

As $(a^3 - b^3)$ is also a prime number or product of prime numbers, hence obviously k = (a – b) ≠ 3 i.e. 3 cannot be a factor of k.

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Because in this case $(a^3 - b^3)$ always contains a factor 3^2 Also, $k \neq 5$ i.e. 5 cannot be a factor of k as shown earlier. Let us extract few examples.

$3^3.(4^3 - 3^3) = 3^3.37$ produces significant R_f as square integer i.e. $12(37.3) - 3.(3^2)^2 = 33^2 \Rightarrow$ roots are $1/6[3.9 \pm 33] = 10, -1$
Hence, $(3.4)^3 - (3.3)^3 = 10^3 - 1^3 \Rightarrow 12^3 + 1^3 = 10^3 + 9^3$
Same relation can be established by other way also.
$2^3.(6^3 - 5^3) = 2^3.(7.13)$ produces significant R_f as $12(7.13) - 3.(2^3)^2 = 30^2$
$\Rightarrow$ roots are $1/6[3.8 \pm 30] = 9, -1$
Hence, $(2.6)^3 - (2.5)^3 = 9^3 - 1^3 \Rightarrow 12^3 + 1^3 = 10^3 + 9^3$

Let us take another example.

$3^3.(9^3 - 8^3) = 3^3.(7.31)$ produces significant R_f as $12(7.31.3) - 3.(3^2)^2 = 87^2$
$\Rightarrow$ roots are $1/6[3.9 \pm 87] = 19, -10$
Hence, $(3.9)^3 - (3.8)^3 = 19^3 - 10^3 \Rightarrow 27^3 + 10^3 = 24^3 + 19^3$
Same relation can be established by other way also.
$2^3.(12^3 - 5^3) = 2^3.(7.229)$ produces significant R_f as $12(7.229) - 3.(2^3)^2 = 138^2$
$\Rightarrow$ roots are $1/6[3.8 \pm 138] = 27, -19$
Hence, $(2.12)^3 - (2.5)^3 = 27^3 - 19^3 \Rightarrow 24^3 + 19^3 = 10^3 + 27^3$

Let us take another interesting example:

$3^3(3^3 + 5^3) = 2^3\{3^3(19)\} = 2^3(8^3 + 1^3) = 16^3 + 2^3$ where significant R_f for $3^3.(19)$ is $12(19.3) - 3.9^2 = 21^2$ & corresponding roots are 8, 1
This interchanging phenomenon happens when both the elements of $3^3.(a^3 + b^3)$ are odd.

5. Analysis of Ramanujan relations of higher exponents by simple algebraic equation.

Say, $N = xy = \alpha^{2n+1} + \beta^{2n+1} = (\alpha + \beta)[\alpha^{2n} - \alpha^{2n-1}\beta + \alpha^{2n-2}\beta^2 - \ldots\ldots\ldots \beta^{2n}]$ where xy is the product of prime numbers only & $\alpha + \beta = x$.
$\Rightarrow \alpha^{2n} - \alpha^{2n-1}(x - \alpha) + \alpha^{2n-2}(x - \alpha)^2 - \ldots\ldots\ldots + (x - \alpha)^{2n} = y$
$\Rightarrow (2n + 1)\alpha^{2n} - x(k_1)\alpha^{2n-1} + x^2(k_2)\alpha^{2n-2} - \ldots\ldots\ldots (x^{2n} - y) = 0$ i.e. $f(\alpha) = 0$ where $|\alpha| \geq 1/(2n + 1)$ and
It will produce pair wise integer or fractional roots like $(\alpha_1, x - \alpha_1), (\alpha_2, x - \alpha_2), (\alpha_3, x - \alpha_3), \ldots\ldots$ to have R-relations
$\alpha_1^{2n+1} + (x - \alpha_1)^{2n+1} = \alpha_2^{2n+1} + (x - \alpha_2)^{2n+1} = \alpha_3^{2n+1} + (x - \alpha_3)^{2n+1} = \ldots\ldots\ldots\ldots$
If all the roots are imaginary or irrational or fractional the number fails to produce even a single wing. For a single wing there must exist only one pair of integer roots.
Now, $f(\alpha/p) = 0$ will produce roots $p\alpha_1, p\alpha_2, p\alpha_3, \ldots\ldots$ So, in the equation $f(\alpha) = 0$ if there is any fractional root of the form α_k/p it will be modified as integer roots like $p(\alpha_k/p) = \alpha_k$.
Then we will get R-relation like $(p\alpha_1)^{2n+1} + \{p(x - \alpha_1)\}^{2n+1} = (p\alpha_2)^{2n+1} + \{p(x - \alpha_2)\}^{2n+1} = \ldots\ldots\ldots = \alpha_k^{2n+1} + (px - \alpha_k)^{2n+1}$ which can be considered as additional wing. This type of additional wing can be produced more also as because sum of the roots is an integer ($= nx$) and that is why, fractional roots cannot exist alone. It comes pair wise & each pair create a coincident wing e.g. for exponent 3, if the fractional roots are $x - \alpha/3$ & $\alpha/3$ coincident wing is $(x - \alpha/3)^3 + (\alpha/3)^3$ or $\{x - (x - \alpha/3)^3 - (x - \alpha/3)^3$ and after multiplication of 3^3 we get a single significant wing $(3x - \alpha)^3 + \alpha^3$.
Similar pair wise fractional roots' phenomenon along with same pair wise integer roots' phenomenon can occur several times for higher exponents or many more factors of N so as to create more additional significant wings along with the group of insignificant wings. But again by replacement theory $f(\alpha/p^2) = 0$ cannot produce any significant roots except p times of earlier roots. Hence, no significant wing is possible to be created.
$\Rightarrow f(\alpha/p^m) = 0$ for $m > 1$ cannot produce any significant relation.
So, a Ramanujan Number (RN) of any accepted exponent may consist of two groups: one is of significant groups where all expressions are free from any common factor and other is of insignificant groups where all expressions have a common factor. If we view only the insignificant group it is actually a group of under significant group only. The significant group cannot exist unless insignificant group exist.
Now, if $a^5 + b^5 = c^5 + d^5$ exists there is no existence of $a^{5\alpha} + b^{5\alpha} = c^{5\alpha} + d^{5\alpha}$ for $\alpha > 1$ due to same logic given in earlier heading 3.
Here, there must exist two elements which are divisible by 5 & 2. & a^α or d^α cannot contain any factor only 5 or only 2. So the relation can be written as $5^{5(\alpha-1)}\{(5a^\alpha)^5 - (5c^\alpha)^5\}$ which is meaningless for $\alpha > 1$ and $\neq d^{5\alpha} - b^{5\alpha}$ Because in this insignificant case d & b both must have a factor $5^{5(\alpha-1)}$ which is quite impossible.
Similarly, $2^{5(\alpha-1)}\{(5d^\alpha)^5 - (5b^\alpha)^5\}$ is also meaningless for $\alpha > 1$ and $\neq a^{5\alpha} - c^{5\alpha}$.
On the same logic, if $a^7 + b^7 = c^7 + d^7$ exists there is no existence of $a^{7\alpha} + b^{7\alpha} = c^{7\alpha} + d^{7\alpha}$ for $\alpha > 1$ & so on.
Earlier we have seen by N_d/N_s operation that $a^{2\alpha} + b^{2\alpha} = c^{2\alpha} + d^{2\alpha}$ has also no existence for $\alpha > 1$

[Refer papers published in Aug-edition 2013, vol-4 of IJSER]

Hence, we can conclude that if $a^n + b^n = c^n + d^n$ exists it will exist only for prime number of n.
$\Rightarrow$ p = either n or 2 as shown under heading 2. Any relation can be established by following three ways.
$n^n.(a^n - b^n)$ where a, b are the combinations of even and odd.
$2^n.(a^n - b^n)$ where a, b are the combinations of even and odd.
$(2n)^n(a^n - b^n)$ where a, b are the odd integers.

Note: $N = a^n + b^n$ where gcd(a, b) = 1 does not contain any factor p^m where $m > n$

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Conclusion:

In a nutshell we can say that if $a^n + b^n = c^n + d^n$ where a, b, c, d all are non-zero integers, exists n must be a prime number and there must exist at least one pair of elements with common factor either 2 or n or both. I believe that all the logics and analysis given for the acceptance of Ramanujan exponents and the way to recognize a number whether it is capable of producing R-relations under exponent 3 or not, will be accepted by all mathematical communities. Now the most difficult task is to arrange all R-examples under a particular exponent in a systematic manner as it has been possible to do so in case of $a^2 + b^2 = c^2$. I also believe that with the help of this theory it is possible to analyze the case of variable exponents. On the way of achieving these two targets the hurdle that we have to face is that we do not have any generalized method to solve an algebraic equation of single variable having degree more than 4. However, problems and solutions both are co-existent, that's why research activities are alive.

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I have already introduced myself as a person who is not a mathematician by profession but a civil engineer passed from Bengal Engineering College under then Calcutta University in 1980, presently working as Sr. Manager in a Public Sector Oil Company posted at Bihar. I am born and brought up at Kolkata (earlier Calcutta). I always try to utilize my free time by different mathematical games particularly in Number theory. Number theory attracts me more. Because, it is such a field where most of the things seem to be quite natural or obvious, but it is very difficult to prove. To me, every number is full of mysteries like a star in the sky.
Being a person not in the proper field of mathematics or research work, initially I faced lot of difficulties regarding preparation of manuscript and its publication. IJSER helped me a lot by their laid down instructions and guidance. I am really grateful to IJSER that they have published my papers and brought my thoughts to the knowledge of all mathematical communities.